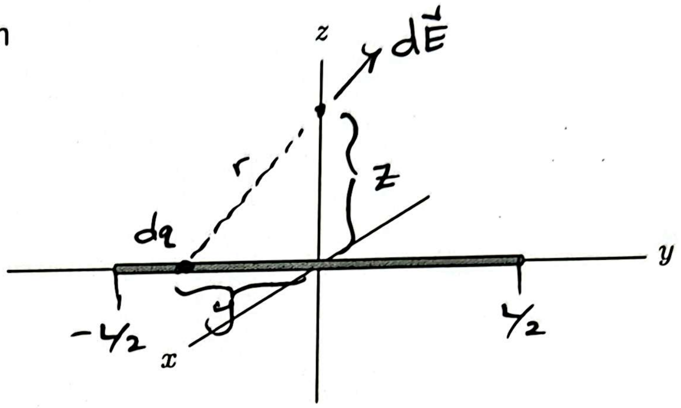
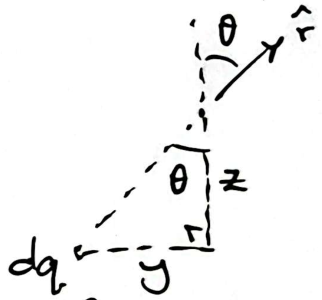
Find the electric field a distance z away from the center of a line charge (with charge density λ C/m).



$$\vec{E} = \int d\vec{E} = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\hat{r}$$

in terms of y: $dq = dy \cdot \lambda$

$$r^2 = y^2 + z^2$$



$$\hat{r} = \hat{j}\sin\theta + \hat{k}\cos\theta$$

$$\sin\theta = \frac{-y}{\sqrt{y^2+z^2}} \qquad \cos\theta = \frac{z}{\sqrt{y^2+z^2}}$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{dy}{y^2+z^2}\cdot\left(\underbrace{\cancel{-\hat{j}\frac{y}{\sqrt{y^2+z^2}}}}_{\text{drop this}} + \hat{k}\frac{z}{\sqrt{y^2+z^2}}\right)$$

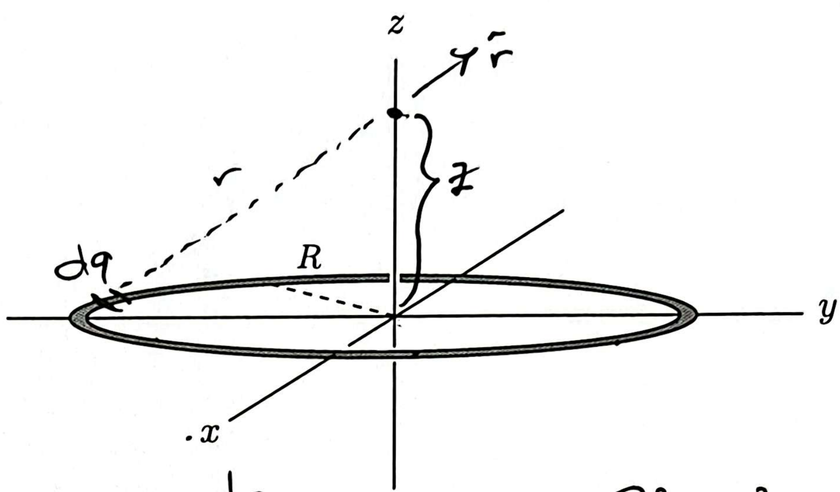
$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\hat{k} z \int_{-L/2}^{L/2} \frac{dy}{(y^2+z^2)^{3/2}} = \frac{\lambda}{4\pi\epsilon_0} z\hat{k}\left[\frac{y}{z^2\sqrt{y^2+z^2}}\right]_{-L/2}^{L/2}$$

$$= \frac{\lambda z \hat{k}}{4\pi\epsilon_0}\cdot\frac{L/2 \cdot 2}{z^2\sqrt{\left(\frac{L}{2}\right)^2+z^2}} = \frac{\lambda\hat{k}}{4\pi\epsilon_0}\frac{L}{z\sqrt{(L/2)^2+z^2}}$$

$Q = \lambda L$

$$\boxed{= \frac{1}{4\pi\epsilon_0}\frac{Q}{z\sqrt{(L/2)^2+z^2}}\hat{k}}$$

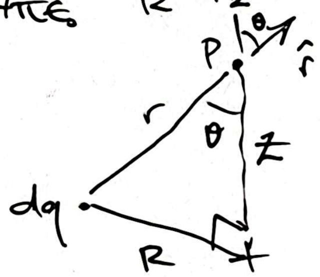
Find the field along the z axis due to a ring of line charge (charge density $\underline{\underline{\lambda}}$ C/m)

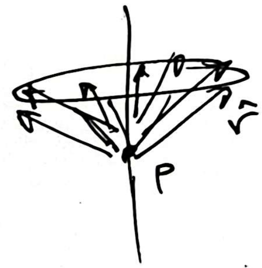


$$\vec{E} = \int d\vec{E} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}\hat{r}$$

$r^2 = R^2 + z^2$
(constant)

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{R^2+z^2} \cdot \int dq \cdot \hat{r}$$





horizontal components cancel!

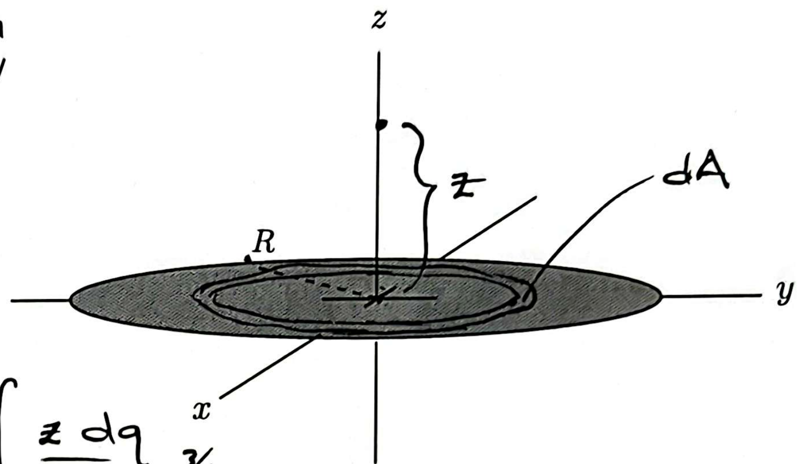
$$\hat{r} = (\cos\vartheta)\,\hat{k} + \cancel{(\hat{\imath}, \hat{\jmath})}$$

$$\cos\vartheta = \frac{z}{\sqrt{R^2+z^2}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{1}{(R^2+z^2)} \frac{z}{\sqrt{R^2+z^2}}\,\hat{k} \overbrace{\int dq}^{Q}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{zQ}{(R^2+z^2)^{3/2}}\,\hat{k}}$$

Find the field along the z axis due to a disk of surface charge (charge density σ C/m^2)



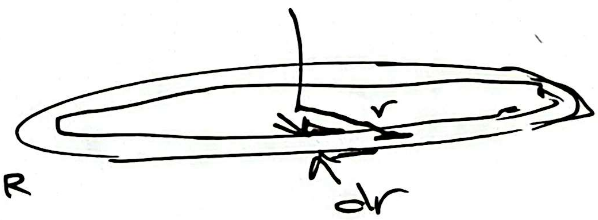
$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\epsilon_0}\hat{k}\int \frac{z\,dq}{(r^2+z^2)^{3/2}}$$

field of a ring charge

$$dq = \sigma\, dA \qquad A = \pi r^2 \qquad \frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r \cdot dr$$

$$dq = \sigma \cdot 2\pi r \cdot dr$$



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\hat{k}\cdot z\cdot\sigma\cdot 2\pi \int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

$$= \frac{1}{2\epsilon_0}\cdot z\sigma\hat{k} \cdot \underbrace{\left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^R}$$

$$= \frac{z\sigma}{2\epsilon_0}\hat{k}\cdot\left(-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z}\right)$$

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{k}}$$

How does this field behave when $z \gg R$?

if $z \gg R$ $\quad \sqrt{R^2+z^2} \approx \sqrt{z^2} = z$

then $\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{z}\right)\hat{k} = 0$

Sure, it goes to zero, but we want to know <u>how</u> it goes to zero.

To do this we can use the "binomial approximation", which says that

$$(1+x)^\alpha \approx 1 + \alpha x \quad \text{if } x \ll 1.$$

(This is really just the first two terms of the Taylor series.)

Consider the second term in the parentheses:

$$\frac{z}{\sqrt{R^2+z^2}} \cdot \frac{1/z}{1/z} = \frac{1}{\sqrt{\left(\frac{1}{z}\right)^2(R^2+z^2)}} = \frac{1}{\sqrt{\left(\frac{R}{z}\right)^2+1}} = \left[1+\left(\frac{R}{z}\right)^2\right]^{-1/2}$$

This has the form of the binomial above, with $x = (R/z)^2$. Clearly $\left(\frac{R}{z}\right)^2 \ll 1$. So:

$$\frac{z}{\sqrt{R^2+z^2}} = \left(1+\left(\frac{R}{z}\right)^2\right)^{-1/2} \approx 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2$$

$$\vec{E} \approx \frac{\sigma}{2\epsilon_0}\left(1 - 1 + \frac{1}{2}\frac{R^2}{z^2}\right)\hat{k} = \frac{\sigma}{4\epsilon_0}\frac{R^2}{z^2}\hat{k}$$

with $\sigma = \frac{Q}{\pi R^2}$, this is $\quad \vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{z^2}\hat{k}$ QED