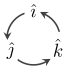
ENGINEERING MECHANICS DYNAMICS

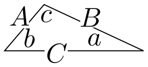
MADISON AREA | TECHNICAL COLLEGE

MATH

$$\vec{A}\cdot\vec{B} = AB\cos\theta \;= A_xB_x + A_yB_y + A_zB_z \qquad \hat{u} = \vec{A}/A$$

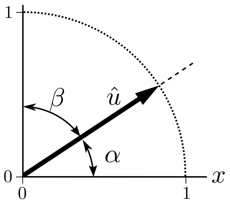


$$\vec{A}\times\vec{B} = (AB\sin\theta)\,\hat{u}_\perp \;= (A_yB_z - A_zB_y)\,\hat{\imath} + (A_zB_x - A_xB_z)\,\hat{\jmath} + (A_xB_y - A_yB_x)\,\hat{k}$$



$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad C^2 = A^2 + B^2 - 2AB\cos c$$

$$\hat{u} = \hat{\imath}\cos\alpha + \hat{\jmath}\cos\beta$$
$$= \hat{\imath}\cos\alpha + \hat{\jmath}\sin\alpha$$



PARTICLE KINEMATICS

Curvilinear 1D Motion

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} \qquad a\;ds = v\;dv$$

constant acceleration

$$v(t) = v_0 + at \qquad s(t) = s_0 + v_0t + \tfrac{1}{2}at^2 \qquad v^2 = v_0^2 + 2a\Delta s$$

Cartesian Coordinates

$$v_x = \dot{x} \qquad a_x = \ddot{x}$$
$$v_y = \dot{y} \qquad a_y = \ddot{y}$$
$$v_z = \dot{z} \qquad a_z = \ddot{z}$$

n,t,b Coordinates

$$a_b = 0$$
$$v = \dot{s} \qquad a_n = \frac{v^2}{\rho} \qquad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$
$$a_t = \dot{v} = v\frac{dv}{ds} \qquad \tan\psi = \pm\frac{r}{dr/d\theta}$$

Cylindrical Coordinates

$$v_r = \dot{r} \qquad a_r = \ddot{r} - r\dot{\theta}^2$$
$$v_\theta = r\dot{\theta} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$v_z = \dot{z} \qquad a_z = \ddot{z}$$

WORK & ENERGY

Increment of work done by a force moving along a path increment $d\vec{s}$:

$$dU = \vec{F}\cdot d\vec{s}$$

Work Energy Theorem

$$\sum U_{1\to2} = \Delta T$$

The work done by all forces acting on and within a system evolving from state 1 to state 2 is equal to the system's change in kinetic energy.

Work done by conservative forces may be expressed as a potential energy change.

$$U_{\text{cons}} = -\Delta V$$
$$U_{\text{noncons}} = \Delta(T+V)$$

if no noconservative forces: $\Delta(T+V) = 0$

Kinetic Energy

particle: $T = \frac{1}{2}mv^2$

rigid body in planar (2D) motion:

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2$$

if fixed point P: $T = \frac{1}{2}I_P\omega^2$

rigid body in general (3D) motion:

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}\vec{H}_G\cdot\vec{\omega}$$

Potential Energy

$$V_g = mgy$$
$$V_e = \tfrac{1}{2}ks^2$$

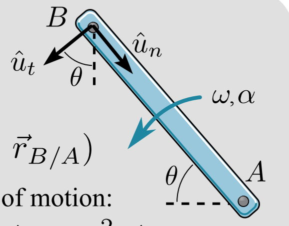
Power & Efficiency

$$P = \frac{dU}{dt} = \vec{F}\cdot\vec{v}$$
$$\varepsilon = \frac{P_{\text{out}}}{P_{\text{in}}} < 1$$

ROTATION & RELATIVE MOTION

$$\vec{v}_{A/B} = -\vec{v}_{B/A} \qquad \vec{a}_{A/B} = -\vec{a}_{B/A}$$
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

If A and B are on the same rigid body, their relative motion is pure rotation:



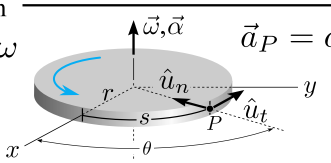
$$\vec{v}_{B/A} = \vec{\omega}\times\vec{r}_{B/A}$$
$$\vec{a}_{B/A} = \vec{\alpha}\times\vec{r}_{B/A} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{B/A})$$

If, additionally, A and B are in the plane of motion:

$$\vec{v}_{B/A} = \omega r\hat{u}_t \qquad \vec{a}_{B/A} = \alpha r\hat{u}_t + \omega^2r\hat{u}_n$$

Circular Motion

$$\alpha\;d\theta = \omega\;d\omega$$
$$\omega = \dot{\theta}$$
$$\alpha = \dot{\omega}$$

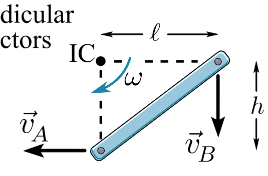


$$\vec{a}_P = \alpha r\hat{u}_t + \omega^2r\hat{u}_n$$
$$s = \theta r$$
$$\vec{v}_P = \omega r\hat{u}_t$$

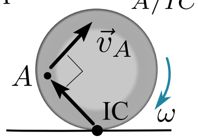
Instantaneous Center of Rotation (IC)

The IC is perpendicular to all velocity vectors on the body.

$$v_A = \omega h$$
$$v_B = \omega\ell$$



$$\vec{v}_A = \vec{\omega}\times\vec{r}_{A/IC}$$



Roll without slip

Rotating Coordinate System *xyz*

With a rotating coordinate system, A and B need not be on the same rigid body. System *xyz* rotates at rate $\vec{\Omega}$.

$$\vec{v}_{B/A} = \vec{\Omega}\times\vec{r}_{B/A} + (\vec{v}_{B/A})_{xyz}$$
$$\vec{a}_{B/A} = \dot{\vec{\Omega}}\times\vec{r}_{B/A} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}_{B/A}) + 2\,\vec{\Omega}\times(\vec{v}_{B/A})_{xyz} + (\vec{a}_{B/A})_{xyz}$$

If, additionally, A and B are in the plane of motion (with $r = |\vec{r}_{B/A}|$):

$$\vec{v}_{B/A} = \Omega r\hat{u}_t + (\vec{v}_{B/A})_{xyz}$$
$$\vec{a}_{B/A} = \dot{\Omega}r\hat{u}_t + \Omega^2r\hat{u}_n + 2\vec{\Omega}\times(\vec{v}_{B/A})_{xyz} + (\vec{a}_{B/A})_{xyz}$$

Time derivative of any vector in the rotating coord. system *xyz* :

$$\dot{\vec{A}} = \left(\dot{\vec{A}}\right)_{xyz} + \vec{\Omega}\times\vec{A}$$

Angular velocities add:

$$\vec{\omega}_{\text{total}} = \sum\vec{\omega}_i$$

IMPULSE & MOMENTUM

linear momentum of a particle: $m\vec{v}$

linear momentum of a system of particles: $\sum_i m_i\vec{v}_i = \left(\sum_i m_i\right)\vec{v}_G$

angular momentum of a particle about point *P*: $\vec{H}_P = \vec{r}\times m\vec{v}$

Angular Momentum of a rigid body *

about center of mass *G*: $\vec{H}_G = I_G\vec{\omega}$

about any point *P*:

$$\vec{H}_P = \vec{r}_{G/P}\times(m\vec{v}_P) + I_P\vec{\omega}$$

$$\vec{H}_P = \vec{r}_{G/P}\times(m\vec{v}_G) + I_G\vec{\omega}$$

if *P* is a fixed point or IC: $\vec{H}_P = I_P\vec{\omega}$

Principle of Impulse and Momentum

$$\int_{t_1}^{t_2}\left(\sum\vec{F}\right)\,dt = \Delta(m\vec{v}) \qquad \int_{t_1}^{t_2}\left(\sum\vec{M}_P\right)\,dt = \Delta\vec{H}_P$$

For a rigid body or a system of particles:
- use only *external* forces and moments
- velocity is $\vec{v}_G$

Conservation of Momentum

with no external forces: $\Delta\left(\sum_i m_i\vec{v}_i\right) = 0$

$\vec{v}_G =$ constant

with no external moments about *P*: $\Delta\left(\sum_i \vec{H}_{Pi}\right) = 0$

Steady Fluid Flow

Identify a volume with unchanging mass. Mass flows in and out at a rate of *dm/dt*, which is constant everywhere along the flow.

$$\frac{dm}{dt} = \rho vA = \rho Q$$

Impuse-Momentum

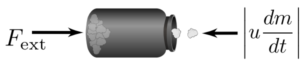
$$\begin{cases}\sum\vec{F} = (\rho Q\,\vec{v})_{\text{out}} - (\rho Q\,\vec{v})_{\text{in}}\\ \sum\vec{M}_O = (\vec{r}\times\rho Q\,\vec{v})_{\text{out}} - (\vec{r}\times\rho Q\,\vec{v})_{\text{in}}\end{cases}$$

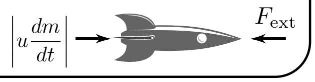
Variable Mass Systems

For systems that change mass, draw the FBD with an extra force of *u dm/dt* to account for mass expulsion or mass accumulation (at speed *u* relative to the device).

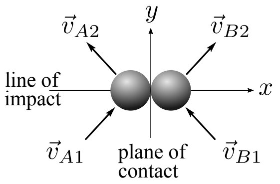
$$F_{\text{ext}} \pm \left|u\frac{dm}{dt}\right| = ma$$

(mass will be a function of time)





Impacts



Define the *x* axis along line of impact, and *y* axis in the plane of contact. Assume smooth bodies.

no impulse in the *y* direction:

$$\begin{cases}v_{A1y} = v_{A2y}\\ v_{B1y} = v_{B2y}\end{cases}$$

coefficient of restitution (*e* = 1 for elastic collition)

$$e = -\frac{v_{B2x} - v_{A2x}}{v_{B1x} - v_{A1x}}$$

If object *A* bounces off a fixed surface (wall), use these equations with $\vec{v}_{B1} = \vec{v}_{B2} = \vec{0}$

x-momentum conserved: $m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$

KINETICS

Equations of Motion *

$$\sum\vec{F} = m\vec{a}_G \qquad \sum\vec{M}_G = I_G\vec{\alpha}$$

all of the following are equivalent:

$$\sum\vec{M}_P = \vec{r}_{G/P}\times(m\vec{a}_P) + I_P\vec{\alpha}$$

$$\sum\vec{M}_P = \vec{r}_{G/P}\times(m\vec{a}_G) + I_G\vec{\alpha}$$

$$\sum\vec{M}_P = I_P\vec{\alpha} \quad \text{if } P \text{ is a fixed point}$$

Free Body Diagram

- Decide exactly what body to analyze.
- Cut this body free from its environment. Draw a closed line (or surface) that completely encircles the body.
- Sketch the body alone.
- Sketch all forces on the body:
  - Gravity (if the body has weight)
  - Wherever the cut passes through a point of contact, draw the forces that replace that contact.
- Label each force with a scalar variable.
- Sketch a coordinate system to the side; show dimensions and angles if helpful.
- Sketch the direction of the center-of-mass acceleration $\vec{a}_G$ and rotational motion.

MOMENT OF INERTIA

Moment of Inertia about axis *a*:

$$I_a = \int_m \underset{\text{distance of } dm \text{ to axis}}{r^2}\ \underset{\rho dV}{dm} = \underset{\text{total mass}}{m}\,\underset{\text{radius of gyration}}{k^2} = \overbrace{I_G + md^2_{G/a}}^{\text{parallel axis theorem}}$$

Products of Inertia

$$I_{xy} = \int_m xy\,dm$$

$$I_{yz} = \int_m yz\,dm$$

$$I_{xz} = \int_m xz\,dm$$

for a body whose mass is symmetric about either the *x*=0 or *y*=0 planes: $I_{xy} = 0$

Inertia Tensor

$$\{I\} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz}\\ -I_{yx} & I_{yy} & -I_{yz}\\ -I_{zx} & -I_{zy} & I_{zz}\end{pmatrix}$$

If the *x-y-z* coordinate axes are oriented along the body's **principal axes**, the inertia tensor is diagonal (products of inertia are zero).
In this case:

$$\vec{H} = \{I\}\,\vec{\omega} = I_x\omega_x\hat{\imath} + I_y\omega_y\hat{\jmath} + I_z\omega_z\hat{k}$$

If the moment of inertia is known about axes with origin *O*, the inertia about an axis in the direction of line *a* given by $\hat{u}_a = u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{k}$ is:

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{xz}u_xu_z$$

Parallel Plane Theorem

$$I_{xy} = (I_{x'y'})_G + mx_Gy_G$$

$$I_{yz} = (I_{y'z'})_G + my_Gz_G$$

$$I_{zx} = (I_{z'x'})_G + mx_Gz_G$$

* For general 3D motion, the moment of inertia must be the inertia tensor $\{I\}$.