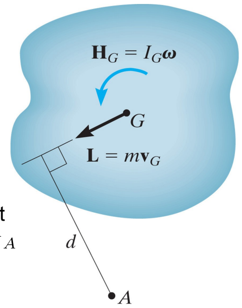
21.2 Angular Momentum

- For planar motion, $\vec{\omega} = \omega\hat{k}$, $\vec{H}_A = H_A\hat{k}$, and I_A is a scalar.

  In that case (text section 19.1), angular momentum was defined as

  $$\vec{H}_A = \vec{r}_{G/A} \times (m\vec{v}_G) + \vec{H}_G$$
  $$\vec{H}_G = I_G\vec{\omega}$$
  or $\vec{H}_O = I_O\vec{\omega}$ for a fixed point O.



- For general 3D motion, $\vec{\omega}$ and $\vec{H}_A$ can point in any direction (3 components each), and $\{I\}_A$ is a tensor (with 6 independent components).

21.2 Angular Momentum

- For 3D motion, using the definition of angular momentum, one can show (see section 21.2) that

$$\vec{H}_A = \vec{r}_{G/A} \times (m\vec{v}_G) + \vec{H}_G$$
$$\vec{H}_G = \{I\}_G\,\vec{\omega}$$
or $\vec{H}_O = \{I\}_O\,\vec{\omega}$ for a fixed point O.

} same as 2D, but now with tensor multiplication

$$\{I\}\,\vec{\omega} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{pmatrix}$$

(matrix multiplication of linear algebra)

21.2 Angular Momentum

A tensor times a vector yields a vector:

$$\vec{H} = \{I\}\,\vec{\omega} = \begin{pmatrix} I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

this is shorthand for 3 equations:

$$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

21.2 Angular Momentum

If the xyz axes align with the principal axes of the body, then all products of inertia are zero ($I_{xy} = I_{xz} = I_{yz} = 0$).

Then
$$\vec{H} = \{I\}\,\vec{\omega} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

or $\vec{H} = I_x\omega_x\hat{\imath} + I_y\omega_y\hat{\jmath} + I_z\omega_z\hat{k}$

Note that $\vec{H}$ does not generally point in the direction of $\vec{\omega}$.

21.3 Kinetic Energy

- For planar motion, $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$
- For general 3D motion, same but use the inertia tensor $\{I\}_G$

  to do this, use $\vec{H}_G = \{I\}_G\,\vec{\omega}$

  as follows: $T = \frac{1}{2}mv_G^2 + \frac{1}{2}\vec{H}_G \cdot \vec{\omega}$
- If the body's principal axes are used, then $\vec{H}_G = I_x\omega_x\hat{\imath} + I_y\omega_y\hat{\jmath} + I_z\omega_z\hat{k}$

  so $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$
- About a fixed point O:

  $T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$ (inertia calculated about O)

21.2-3 3D Kinetics

Angular Momentum:

$$\vec{H}_A = \vec{r}_{G/A} \times (m\vec{v}_G) + \vec{H}_G$$
$$\vec{H}_G = \{I\}_G\,\vec{\omega}$$
$\vec{H}_O = \{I\}_O\,\vec{\omega}$ (fixed point O)

Principle of Impulse & Momentum:

$$\sum \int \vec{F}\,dt = \Delta(m\vec{v}_G)$$
$$\sum \int \vec{M}_A\,dt = \Delta\vec{H}_A$$

Kinetic Energy:

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}\vec{H}_G \cdot \vec{\omega}$$

along principal axes:

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

about fixed point O :

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

Principle of Work & Energy:

$$\sum U_{1-2} = \Delta T$$