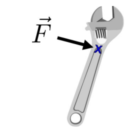
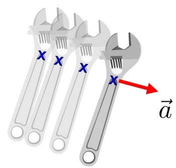
17.2 Planar Kinetic Equations of Motion

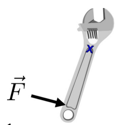
Consider a rigid body subjected to a single force.



If the force is applied at the body's center of mass (G), it accelerates in the direction of the force.

$$\vec{F} = m\vec{a}_G$$

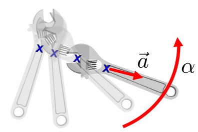




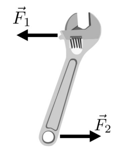
If the force is not applied at G, the acceleration a_G ***is the same***.

In addition, there is angular acceleration α :

$$\vec{M}_G = I_G\vec{\alpha}$$



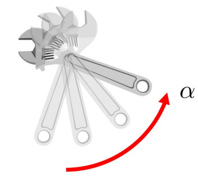
17.2 Planar Kinetic Equations of Motion

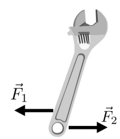


If a **force couple** is applied, there is only rotation.

$$\sum \vec{F} = m\vec{a}_G = 0$$

$$\sum \vec{M}_G = I_G\vec{\alpha}$$





The moment of a force couple is a "free vector" (does not matter where it is applied).

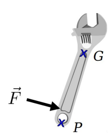
17.2 Planar Kinetic Equations of Motion

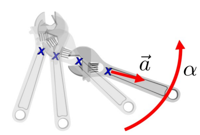
A complication….

What if the moment is taken about a different point, P ?

$$\vec{M}_P \neq \vec{M}_G = I_G\vec{\alpha}$$

The body's motion ($\vec{a}$,$\vec{\alpha}$) cannot depend on how we do the math.





In Statics, we solved $\sum \vec{F} = \vec{0}$ and $\sum \vec{M}_P = \vec{0}$ with moments taken about any point P you chose.

In Dynamics, we solve $\sum \vec{F} = m\vec{a}_G$ and $\sum \vec{M}_P = ?$

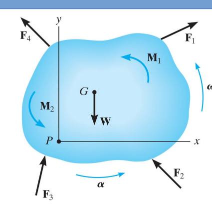
We must go back to Newton's 2nd Law to derive the equations of motion.

17.2 Planar Kinetic Equations of Motion

For a rigid body subjected to several forces and moments, all in the same plane:

- The net force causes whole-body translation:

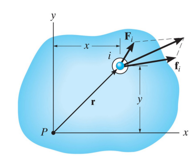
$$\sum \vec{F} = m\vec{a}_G \qquad \text{(proved in section 13.3)}$$



- For the net moment about point P, consider mass element i at location $\vec{r}_i$:

$$\vec{F}_i = m_i\vec{a}_i \longrightarrow \vec{r}_i \times \vec{F}_i = \vec{r}_i \times m_i\vec{a}_i$$

sum of all external and internal forces

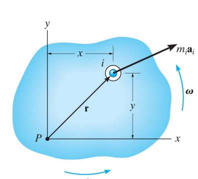


17.2 Planar Kinetic Equations of Motion

$$\vec{M}_{P,i} = \vec{r}_i \times \vec{F}_i = \vec{r}_i \times m_i\vec{a}_i$$

This is easier to interpret if we write the acceleration relative to point P :

$$\vec{a}_i = \vec{a}_P + \vec{a}_{i/P}$$
$$= \vec{a}_P + \alpha r_i\hat{u}_t + \omega^2 r_i\hat{u}_n$$



Then

$$\vec{M}_{P,i} = \vec{r}_i \times m_i\left(\vec{a}_P + \alpha r_i\hat{u}_t + \omega^2 r_i\hat{u}_n\right)$$

$$\vec{r}_i \times \hat{u}_n = 0$$

$$\vec{r}_i \times \hat{u}_t = r_i\,\hat{k}$$

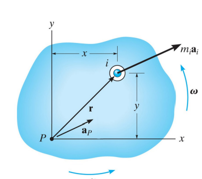
for planar motion

$$= m_i\vec{r}_i \times \vec{a}_P + m_i\alpha r_i^2\hat{k}$$

17.2 Planar Kinetic Equations of Motion

$$\vec{M}_{P,i} = m_i\vec{r}_i \times \vec{a}_P + m_i\alpha r_i^2\hat{k}$$

For total moment on the body, sum over all particles i :



$$\sum_i \vec{M}_{P,i} = \sum_i (m_i\vec{r}_i \times \vec{a}_P + m_i\alpha r_i^2\hat{k})$$

$$\vec{M}_{P,ext} = \left(\sum_i m_i\vec{r}_i\right) \times \vec{a}_P + \left(\sum_i m_i r_i^2\right)\vec{\alpha}$$

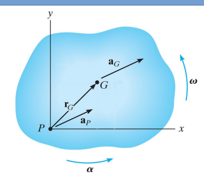
$\vec{\alpha} = \alpha\hat{k}$ for planar motion

total external moment (all internal moments cancel each other)

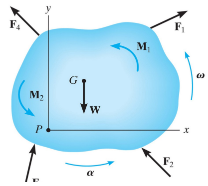
17.2 Planar Kinetic Equations of Motion

$$\vec{M}_P = \underbrace{\left(\sum_i m_i \vec{r}_i\right)}\times \vec{a}_P + \underbrace{\left(\sum_i m_i r_i^2\right)}\vec{\alpha}$$

$$\vec{r}_G = \frac{1}{m}\sum_i m_i \vec{r}_i \qquad I_P = \sum_i m_i r_i^2$$



$$\boxed{\vec{M}_P = \vec{r}_G \times (m\vec{a}_P) + I_P\vec{\alpha}}$$



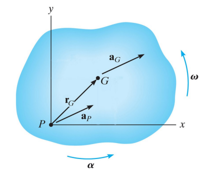
- This result tells us how a rigid body rotates in response to applied moments.
- It is often more useful when expressed in terms of $\mathbf{a}_G$ and I_G. (We do this next.)

17.2 Planar Kinetic Equations of Motion

$$\vec{M}_P = \vec{r}_G \times (m\vec{a}_P) + I_P\vec{\alpha}$$

$I_P = I_G + mr_G^2$ (parallel axis theorem)

$$\vec{a}_G = \vec{a}_P + \vec{a}_{G/P}$$
$$= \vec{a}_P + \alpha r\hat{u}_t + \omega^2 r_G \hat{u}_n$$



$$\vec{M}_P = \vec{r}_G \times m(\vec{a}_G - \alpha r\hat{u}_t - \omega^2 r_G\hat{u}_n) + I_G\vec{\alpha} + mr_G^2\vec{\alpha}$$

$$\vec{M}_P = \vec{r}_G \times (m\vec{a}_G) - mr_G^2\alpha\hat{k} + I_G\vec{\alpha} + mr_G^2\vec{\alpha}$$

$\vec{r}_G \times \hat{u}_n = 0$

$\vec{r}_G \times \hat{u}_t = r_G\hat{k}$ for planar motion

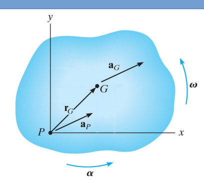
$\vec{\alpha} = \alpha\hat{k}$ for planar motion

17.2 Planar Kinetic Equations of Motion

$\vec{r}_G = \vec{r}_{G/P}$

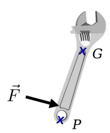
$$\boxed{\vec{M}_P = \vec{r}_G \times (m\vec{a}_G) + I_G\vec{\alpha}}$$

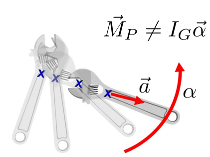
net moment on body about P



This term accommodates arbitrary choice of point P.

- Hibbeler calls it a "kinetic moment" about P : $(\mathcal{M}_k)_P$
- You can think of it as a sort of parallel-axis theorem.

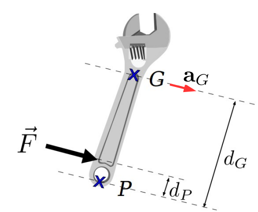




17.2 Planar Kinetic Equations of Motion

$$\boxed{\vec{M}_P = \vec{r}_G \times (m\vec{a}_G) + I_G\vec{\alpha}}$$

for example
$$\begin{cases} \vec{M}_P = -Fd_P\hat{k} \\ \vec{r}_G \times m\vec{a}_G = -d_G\underbrace{ma_G}_{F}\hat{k} \end{cases}$$



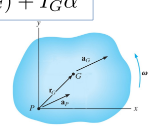
$$-Fd_P\hat{k} = -d_G F\hat{k} + I_G\vec{\alpha}$$

$$F(d_G - d_P)\hat{k} = I_G\vec{\alpha} \longrightarrow \vec{M}_G = I_G\vec{\alpha}$$

17.2 Planar Kinetic Equations of Motion

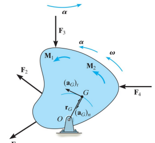
Two equivalent ways to find how a rigid body rotates in response to applied moments:

$$\boxed{\vec{M}_P = \vec{r}_G \times (m\vec{a}_P) + I_P\vec{\alpha}} \qquad \boxed{\vec{M}_P = \vec{r}_G \times (m\vec{a}_G) + I_G\vec{\alpha}}$$



Notes:

- $\vec{r}_G = \vec{r}_{G/P}$
- moments are found about point P, which can be anywhere
- if P is at the CM (G) then $\vec{M}_P = \vec{M}_G = I_G\vec{\alpha}$
- if P is a fixed point (so $\vec{a}_P = 0$) then $\vec{M}_P = I_P\vec{\alpha}$



17.2 Planar Kinetic Equations of Motion

Procedure:

- Draw free-body diagram.
- Apply equations of motion: $\sum \vec{F} = m\vec{a}_G$ and one of
$$\begin{cases} \sum \vec{M}_P = \vec{r}_G \times (m\vec{a}_P) + I_P\vec{\alpha} \\ \sum \vec{M}_P = \vec{r}_G \times (m\vec{a}_G) + I_G\vec{\alpha} \\ \sum \vec{M}_G = I_G\vec{\alpha} \end{cases}$$
- Apply simplifications:
  - If no rotation (pure translation) then $\omega = \alpha = 0$ $\quad a_P = \vec{a}_G$
  - If P is a fixed point : $\vec{a}_P = 0$
- Apply known constraints. For example, relative motion $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ or rolling without slip ($v_G = r\omega$, $a_G = r\alpha$)
- Apply kinematic equations. $\alpha = \dfrac{d\omega}{dt} \qquad \omega = \dfrac{d\theta}{dt} \qquad \alpha d\theta = \omega d\omega$